EXERCISESET 3, TOPOLOGY IN PHYSICS

- The hand-in exercise is the exercise 2.
- Please hand it in electronically at `topologyinphysics2019@gmail.com` (1 pdf!)
- Deadline is Wednesday February 27, 23.59.
- Please make sure your name and the week number are present in the file name.

Exercise 1: Maxwell theory and de Rham cohomology. The advantage of formulating Maxwell's theory in terms of differential forms is that it now makes sense on any manifold M, not even 4-dimensional! For this we consider the first few terms of the de Rham complex:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

As we have seen, the electric and magnetic fields are gathered in a two-form $F \in \Omega^2(M)$, which the homogeneous Maxwell equations require to be closed: $dF = 0$.

a) Assume that $H^2_{\mathrm{dR}}(M) = 0$. Show that for any field strength F there is a "potential" $A \in \Omega^1(M)$ such that $F = dA$. Show that two potentials A and $A + d\Lambda$, with $\Lambda \in C^\infty(M)$ describe the same configuration of the electromagnetic fields, so that the "configuration space" of possible electromagnetic fields satisfying the homogeneous Maxwell equations is given by the quotient $\Omega^1(M)/d\Omega^0(M)$. Elements in $d\Omega^0(M)$ are called "gauge transformations".

b) For any manifold M, we write Ω^k_{cl} for the space of *closed* k-forms. Show that there is a sequence of maps

$$(\star) \qquad 0 \to H^1_{\mathrm{dR}}(M) \to \Omega^1(M)/d\Omega^0(M) \to \Omega^2_{\mathrm{cl}}(M) \to H^2_{\mathrm{dR}}(M) \longrightarrow 0.$$

Explain what are the maps and show that the sequence is exact.

Minkowski space $\mathbb{R}^{1,3}$ is topologically trivial, so $H^1_{\mathrm{dR}}(\mathbb{R}^{1,3}) = 0 = H^2_{\mathrm{dR}}(\mathbb{R}^{1,3})$, and the sequence ($\star$) amounts to an identification $\Omega^2_{\mathrm{cl}}(\mathbb{R}^{1,3}) \cong \Omega^1(\mathbb{R}^{1,3})/d\Omega^0(\mathbb{R}^{1,3})$. In other words: we may equally well describe the electromagnetic field using the potential A, as long as we make sure that we use "gauge invariant" observables, i.e., functions $A \mapsto O(A)$ that are invariant under shifts by $d\Omega^0(\mathbb{R}^{1,3})$: $O(A + d\Lambda) = O(A)$. For a topologically nontrivial manifold M (already $M = \mathbb{R}^3 \times S^1$ is an example) we no longer have $\Omega^2_{\mathrm{cl}}(M) \cong \Omega^1(M)/d\Omega^0(M)$, as the sequence ($\star$) shows. One of the lessons from Quantum Mechanics, as witnessed for example by the Aharonov–Bohm effect is that the potential A, is more fundamental than the field strength F! Therefore, it is better to describe Maxwell theory as a action functional on the space of "fields" $A \in \Omega^1(M)$.

c) Let $\gamma : S^1 \to M$ be a smooth closed curve in M. Show that the function

$$O_\gamma(A) := \int_\gamma A, \quad A \in \Omega^1(M)$$

is a gauge invariant observable. When the field strength $F = 0$, use de Rham's theorem to show that these observables can detect the class in $H^1_{\mathrm{dR}}(M)$.

d) Show that the action functional

$$S(A) = \frac{1}{2}||dA||^2 = \frac{1}{2}\int_M dA \wedge \star dA$$

is gauge invariant and variation leads to the vacuum Maxwell equation $d \star F = 0$. (You actually may have done this already last week...)

$\star$ **Exercise 2: The Hopf fibration.** We consider the 3-sphere defined as

$$S^3 := \{(z_1, z_2 \in \mathbb{C}^2,\ |z_1|^2 + |z_2|^2 = 1\}$$

and recall the definition of the complex projective line $\mathbb{P}^1$ better known as the Riemann sphere

$$\mathbb{P}^1 := \left(\mathbb{C}^2 \backslash \{(0,0)\}\right) / \,\mathbb{C}^\times$$

where $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$ acts by scalar multiplication. Note that S^3 is given by all pairs of complex numbers satisfying a certain equation, while $\mathbb{P}^1$ is given by pairs of complex numbers (z_1, z_1) (not both 0) up to a certain equivalence, namely $(z_1, z_2) \sim (w_1, w_2)$ if there is $0 \neq \lambda \in \mathbb{C}$ such that $\lambda z_1 = w_1$ and $\lambda z_2 = w_2$.

i) The group $U(1) \cong S^1$ acts on S^3 by

$$(z_1, z_2) \cdot e^{i\theta} := (z_1 e^{i\theta}, z_2 e^{i\theta})$$

Find a smooth map $S^3/U(1) \longrightarrow \mathbb{P}^1$ that allows for a smooth inverse, i.e. show that $S^3/U(1) \cong \mathbb{P}^1$.

We may compose the map of $i)$ with the quotient map $S^3 \to S^3/U(1)$ to obtain a map $\pi\colon S^3 \to \mathbb{P}^1$. Recall from the lecture notes of lecture 1 that we had the atlas of $\mathbb{P}^1$ given by the charts

$$U := \{[(z_1, z_2)] \in \mathbb{P}^1 \mid z_1 \neq 0\}$$

and

$$V := \{[(z_1, z_2)] \in \mathbb{P}^1 \mid z_2 \neq 0\}.$$

ii) Find sections $U \to \pi^{-1}(U)$ and $V \to \pi^{-1}(V)$.

iii) Compute the transition function $\varphi_{UV} : U \cap V \to U(1)$.

(BONUS) Consider the standard (defining) representation of $U(1)$ on $\mathbb{C}$:

$$e^{i\theta} \cdot z = e^{i\theta} z,$$

and consider the line bundle associated to the Hopf fibration above. Show that this line bundle agrees with the tautological line bundle over $\mathbb{P}^1$.

Exercise 3: The Hodge–Maxwell Theorem. In this exercise we will define the Hodge $\star$ starting from a general (pesudo-Riemannian) metric g on the oriented manifold M without using coordinates. Recall that g allows us to define a notion of volume on the manifold M. The volume of the submanifold B is given as the integral $\int_B \mathrm{vol}$. In coordinates vol is given by the formula

$$\mathrm{vol} = \sqrt{|g|} dx^1 \wedge \ldots \wedge dx^n,$$

where

$$|g| = \left| \sum_{i_1,\ldots,i_n=1}^{n} \varepsilon_{i_1\ldots i_n} g_{1i_1} \cdots g_{ni_n} \right|$$

denotes the absolute value of the determinant of g and the dx^i form a positively oriented basis. Recall that the coordinate transformation $x^i \to y^i$ is called positive if $\mathrm{Det}\frac{\partial x^i}{\partial y^j}$ (the Jacobian determinant) is positive.

i): Show that the formula for vol above defines an n-form ω. Do this by performing a (positive) coordinate transformation.

ii): The metric g is given by a symmetric non-degenerate bilinear pairing on the tangent spaces

$$(v, w) \mapsto g_{\mu\nu}(x) v^\mu v^\nu,$$

for $v, w \in T_x M$. Show that we get a $C^\infty(M)$-bilinear pairing

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M),$$

where $\mathfrak{X}(M)$ denotes vector fields.

Note that similarly the maps

$$(\alpha, \beta) \mapsto g^{\mu_1\nu_1}(x) \ldots g^{\mu_p\nu_p}(x) \alpha_{\mu_1\ldots\mu_p} \beta_{\nu_1\ldots\nu_p}$$

define a $C^\infty(M)$-bilinear pairing on $\Omega^p(M)$. In fact this is the pairing $\langle \alpha, \beta \rangle$ mentioned in the lecture notes.

iii): Assume that M is compact and show that

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle \omega$$

defines an $\mathbb{R}$-bilinear, symmetric pairing on $\Omega^p(M)$.

iv): Consider $\beta \in \Omega^p(M)$, define $\star\beta$ as the $n - p$ form satifying

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega$$

for all $\alpha \in \Omega^p(M)$ and show that this definition coincides with the coordinate expression given in the lectures.

Hint: Show first that $\star\beta$ is uniquely defined. To do this note that if a form is 0 around every point, then it vanishes globally.

v): Show that the adjoint d^* of the exterior derivative d is given by the formula $\star d \star$.